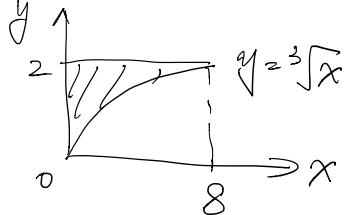


Assignment 1. 15 Sep 2016. Math 2020A

Sec 15.2

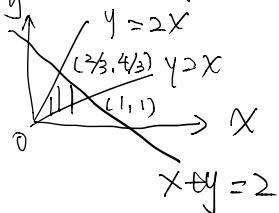
$$54. \int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy dx}{y^4+1} = \int_0^2 dy \int_0^{y^3} \frac{dx}{y^4+1} = \int_0^2 \frac{y^3}{y^4+1} dy$$

$$= \frac{1}{4} [\ln(1+y^4)] \Big|_{y=0}^2 = \frac{1}{4} [\ln 17].$$



$$\{ 0 \leq x \leq 8, \sqrt[3]{x} \leq y \leq 2 \} = \{ 0 \leq y \leq 2, 0 \leq x \leq y^3 \} //$$

55. sketch of R



$$\iint_R xy \, dA = \int_0^{2/3} dx \int_x^{2-x} xy \, dy + \int_{2/3}^1 dx \int_x^{2-x} xy \, dy$$

$$= A + B$$

$$A = \int_0^{2/3} \frac{3}{2} x^3 \, dx = \frac{3}{8} x^4 \Big|_0^{2/3} = \frac{2}{27}$$

$$B = \int_{2/3}^1 \frac{x[(2-x)^2 - x^2]}{2} \, dx = \int_{2/3}^1 2(x-x^2) \, dx$$

$$= (x^2 - \frac{2}{3}x^3) \Big|_{2/3}^1 = 1 - \frac{4}{9} - \frac{2}{3}(1 - \frac{8}{27})$$

$$\text{Integral} = A+B = 13/81, \neq$$

$$64. \iint_{\{|x+y|\leq 1\}} 3(1-x) \, dx \, dy = \int_{-1}^1 dx \int_{-(1-x)}^{1-x} 3(1-x) \, dy = 2 \int_{-1}^1 dx \int_0^{1-|x|} 3(1-x) \, dy$$

$$= 2 \int_0^1 dx \int_0^{1-x} 3(1-x) \, dy + 2 \int_{-1}^0 dx \int_0^{1+x} 3(1-x) \, dy = 6 \int_0^1 (1-x)^2 \, dx + 6 \int_0^1 (1+x)^2 \, dx$$

$$= 6 \neq$$

$$\begin{aligned}
 66. \int_{-\pi/3}^{\pi/3} dx \int_{\sec x}^{\sec x} 1+y^2 dy &= 4 \int_0^{\pi/3} dx \int_0^{\sec x} 1+y^2 dy \\
 &= 4 \int_0^{\pi/3} \left(\sec x + \frac{1}{3} \sec^3 x \right) dx \\
 &= 4 \cdot \left(\frac{1}{12} \ln(1-\sin^2 x) - \frac{1}{12} \left(\frac{1}{\sin x - 1} + \frac{1}{\sin x + 1} \right) \right) \Big|_{x=0}^{\pi/3} = \frac{4\sqrt{3}}{3} - \frac{14}{3} \ln 2
 \end{aligned}$$

where the third " $=$ " is due to the following:

$$\begin{aligned}
 (\sec x + \frac{1}{3} \sec^3 x) dx &\stackrel{t=\sin x \ dt}{=} \frac{dt}{1-t^2} + \frac{1}{3} \frac{dt}{(1-t^2)^2} \\
 &= \frac{1}{2} \left(\frac{1}{1-t} + \frac{1}{1+t} \right) dt + \frac{1}{12} \left(\frac{1}{1-t} + \frac{1}{1+t} \right)^2 dt \\
 &= \left[\frac{1}{2} \left(\frac{1}{1-t} + \frac{1}{1+t} \right) + \frac{1}{12} \left(\frac{1}{(1-t)^2} + \frac{1}{(1+t)^2} + \frac{1}{1-t} + \frac{1}{1+t} \right) \right] dt \\
 &= d \left[\frac{1}{12} \ln(1-t^2) - \frac{1}{12} \left(\frac{1}{t-1} + \frac{1}{t+1} \right) \right]
 \end{aligned}$$

78. Notice $(\tan^{-1} y)' = 1/(1+y^2)$

$$\begin{aligned}
 \int_0^2 \tan^{-1} \pi x - \tan^{-1} x \ dx &= \int_0^2 \int_{\pi x}^{\pi x} \frac{1}{1+y^2} dy \ dx = \int_0^{2\pi} \int_{y/\pi}^{\min\{2, y\}} \frac{1}{1+y^2} dx dy \\
 &= \int_0^2 \int_{y/\pi}^y \frac{1}{1+y^2} dx dy + \int_2^{2\pi} \int_{y/\pi}^2 \frac{1}{1+y^2} dx dy = A + B
 \end{aligned}$$

$$A = \int_0^2 \left(1 - \frac{1}{\pi} \right) \frac{y}{1+y^2} dy = \left(1 - \frac{1}{\pi} \right) \frac{\ln(1+y^2)}{2} \Big|_{y=0}^2 = \left(1 - \frac{1}{\pi} \right) \frac{\ln 5}{2}$$

$$\begin{aligned}
 B &= \int_2^{2\pi} \frac{2}{1+y^2} dy - \int_2^{2\pi} \frac{1}{\pi} \frac{y}{1+y^2} dy \\
 &= 2 \tan^{-1}(y) \Big|_2^{2\pi} - \frac{1}{2\pi} \ln(1+y^2) \Big|_2^{2\pi} \\
 &= 2(\tan^{-1}(2\pi) - \tan^{-1}2) - \frac{1}{2\pi} (\ln(1+4\pi^2) - \ln 5)
 \end{aligned}$$

Thus, Integral = A + B

$$= 2(\tan^{-1}(2\pi) - \tan^{-1}2) - \frac{1}{2\pi} \ln(1+4\pi^2) + \frac{\ln 5}{2} \quad \#$$

Sec. 15.5

$$\begin{aligned}
 30. \text{ Volume} &= \int_0^4 \int_0^{\sqrt{4-y}} \int_0^{4-x^2-y} dz dx dy = \int_0^4 \int_0^{\sqrt{4-y}} 4-x^2-y dx dy \\
 &= \int_0^2 \int_0^{\sqrt{4-x^2}} 4-x^2-y dy dx \\
 &= \int_0^2 (4-x^2)^2 - \frac{1}{2}(4-x^2)^2 dx = \frac{1}{2} \int_0^2 (4-x^2)^2 dx \\
 &= \frac{1}{2} \int_0^2 16 + x^4 - 8x^2 dx \\
 &= \frac{1}{2} (16 \cdot 2 + \frac{1}{5} \cdot 2^5 - \frac{8}{3} \cdot 2^3) = \frac{128}{15} \quad \#
 \end{aligned}$$

$$\begin{aligned}
 32. \text{ Volume} &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{3-x} 1 dz dy dx \\
 &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3-x) dy dx = 2 \int_{-2}^2 \int_0^{\sqrt{4-x^2}} (3-x) dy dx \\
 &= 2 \int_{-2}^2 (3-x) \cdot \sqrt{4-x^2} dx = 6 \int_{-2}^2 \sqrt{4-x^2} dx - 2 \underbrace{\int_{-2}^2 x \sqrt{4-x^2} dx}_{=0 \text{ (ODD Function)}} \\
 &= 24
 \end{aligned}$$

$$t = \frac{x}{2} \Rightarrow 2t = x \quad \text{and} \quad dt = \frac{1}{2} dx \quad \text{so} \quad dx = 2dt$$

$$= 24 \int_{-1}^1 \sqrt{1-t^2} dt = 12\pi$$

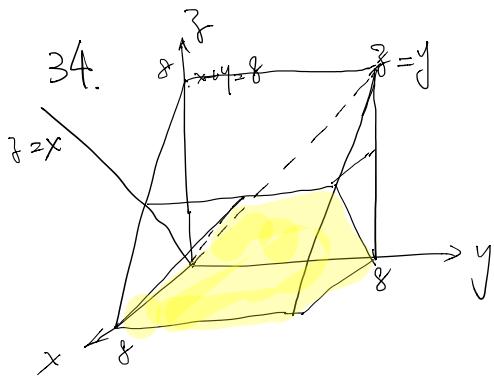
Where the last equality is due to the fact

$$\int_{-1}^1 \sqrt{1-t^2} dt = \frac{1}{2} \cdot \text{area of unit disk} = \frac{1}{2} \cdot \pi$$

or direct calculations as follows.

$$\begin{aligned} \int \sqrt{1-t^2} dt &= \int \sqrt{1-\sin^2 \theta} \cos \theta d\theta = \int \cos^2 \theta d\theta \\ &= \frac{1}{2} \int (\cos 2\theta + 1) d\theta = \frac{1}{4} \sin 2\theta + \frac{1}{2} \theta + C \end{aligned}$$

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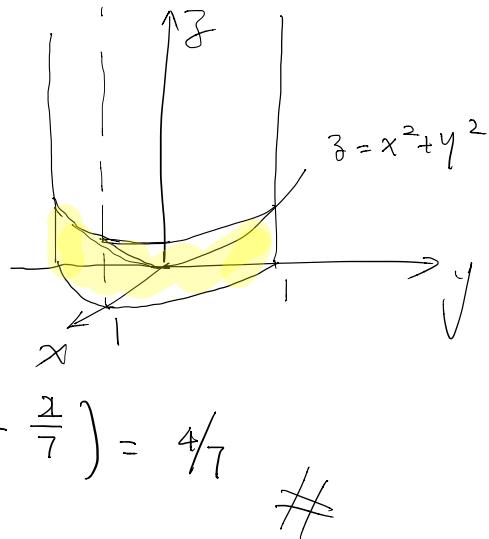
$$\begin{aligned} \text{Volume} &= \int_0^8 \int_8^{8-z} \int_z^8 1 dy dx dz \\ &= \int_0^8 (8-z) \cdot (8-2z) dz \\ &= \int_0^8 64 - 24z + 2z^2 dz \\ &= 64 \cdot 8 - 12 \cdot 8^2 + \frac{2}{3} \cdot 8^3 \\ &\approx 256/3 \end{aligned}$$

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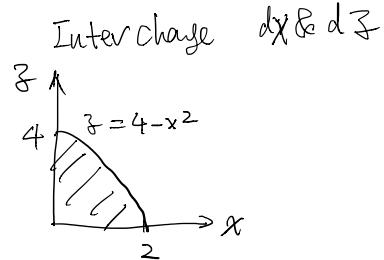
36.

$$\begin{aligned} &\int_{-1}^1 \int_0^{1-y^2} \int_0^{x^2+y^2} 1 dz dx dy \\ &= \int_{-1}^1 \int_0^{1-y^2} x^2+y^2 dx dy \end{aligned}$$

$$\begin{aligned} &= \int_{-1}^1 \frac{1}{3} (1-y^2)^3 + y^2 (1-y^2) dy \\ &= \int_{-1}^1 \frac{1}{3} (1-y^6) dy = \frac{1}{3} \left(2 - \frac{2}{7} \right) = \frac{4}{7} \end{aligned}$$



$$\begin{aligned}
 44. & \int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z} dy dz dx \\
 &= \int_0^4 \int_0^{\sqrt{4-y}} \int_0^y \frac{\sin 2z}{4-z} dy dz dy \\
 &= \int_0^4 \int_0^{\sqrt{4-y}} y \cdot \frac{\sin 2z}{4-z} dy dz \\
 &= \int_0^4 \frac{1}{2} (4-z) \cdot \frac{\sin 2z}{4-z} dz = \frac{1}{2} \int_0^4 \sin 2z dz = \left. \frac{1}{4} (-\cos 2z) \right|_{z=0}^4 \\
 &= \frac{1}{4} (-\cos 8 + \cos 0) = \frac{1}{4} (1 - \cos 8). \quad \text{#}
 \end{aligned}$$



The Addition Exercise

1. i) Let $P = \{(x_{i_1}^1, x_{i_1+1}^1) \times (x_{i_2}^2, x_{i_2+1}^2) \times \dots \times (x_{i_n}^n, x_{i_n+1}^n)\}$

be a partition of A .

Denote $C_{i_1, i_2, \dots, i_n} = (x_{i_1}^1, x_{i_1+1}^1) \times \dots \times (x_{i_n}^n, x_{i_n+1}^n)$

be a cube belonging to P

$$m_{h, i_1, \dots, i_n} = \inf_{x \in C_{i_1, \dots, i_n}} h(x) \quad M_{h, i_1, \dots, i_n} = \sup_{x \in C_{i_1, \dots, i_n}} h(x)$$

for $h = f, g$ and $f+g$.

$$L(f+g, P) = \sum m_{f+g, i_1, \dots, i_n} |C_{i_1, \dots, i_n}|$$

$$\geq \sum (m_{f, i_1, \dots, i_n} + m_{g, i_1, \dots, i_n}) |C_{i_1, \dots, i_n}|$$

$$= \sum m_{f, i_1, \dots, i_n} |C_{i_1, \dots, i_n}| + \sum m_{g, i_1, \dots, i_n} |C_{i_1, \dots, i_n}| = L(f, P) + L(g, P)$$

where the inequality is due to the fact

$$m_{f+g, i_1, \dots, i_n} = \inf_{x \in C_{i_1, \dots, i_n}} f+g \geq \inf_{x \in C_{i_1, \dots, i_n}} f + \inf_{x \in C_{i_1, \dots, i_n}} g$$

$$= m_f, i_1, \dots, i_n + m_g, i_1, \dots, i_n$$

which can be proved by the following: $\forall \varepsilon > 0, \exists x_0 \in C_{i_1, \dots, i_n}$

$$\inf_{x \in C_{i_1, \dots, i_n}} f+g \geq (f+g)(x_0) - \varepsilon = f(x_0) + g(x_0) - \varepsilon$$

$$\geq \inf_{x \in C_{i_1, \dots, i_n}} f + \inf_{x \in C_{i_1, \dots, i_n}} g - \varepsilon$$

1

Similar argument and the fact

$$M_{f+g, i_1, \dots, i_n} \leq M_{f, i_1, \dots, i_n} + M_{g, i_1, \dots, i_n}$$

Can show the other part

$$U(f+g, P) \leq U(f, P) + U(g, P)$$

ii) Let $\{P_n\}$ be a partition sequence and P_{n+1} is a refinement of P_n for $\forall n \geq 1$.

Then $L(f+g, P_n) \leq U(f+g, P_n) \leq U(f, P_n) + U(g, P_n)$

and $L(f+g, P_n) \geq L(f, P_n) + L(g, P_n)$

by taking $n \rightarrow \infty$ and $\begin{matrix} f, g \\ \uparrow \text{the fact} \end{matrix}$ are integrable

$$\int_A f + \int_A g \leq \lim_{n \rightarrow \infty} L(f+g, P_n) \leq \lim_{n \rightarrow \infty} U(f+g, P_n) \leq \int_A f + \int_A g$$

The partition $\{P_n\}$ is chosen as follows.
 (See remark in notes). Let $\{P_n\}, \{P_m\}$ be two partition sequences such that $\lim_{n \rightarrow \infty} L(f, P_n) = \lim_{m \rightarrow \infty} U(f, P_m) = \int_A f$
 $\lim_{n \rightarrow \infty} L(g, P_n) = \lim_{m \rightarrow \infty} U(g, P_m) = \int_A g$. P_n is chosen as
 the refinement of P_{n+k}, P_{n+k} with all $k \in \mathbb{N}$.

thus $f+g$ is integrable and

$$\int_A f+g = \int_A f + \int_A g.$$

iii) Similar to the above argument and the facts

$$M_{cf, i_1, \dots, i_n} = c M_{f, i_1, \dots, i_n}, \quad M_{f, ci_1, \dots, i_n} = c M_{f, i_1, \dots, i_n}$$

for $c \geq 0$ and

$$m_{cf, i_1, \dots, i_n} = c m_{f, i_1, \dots, i_n}, \quad M_{cf, i_1, \dots, i_n} = c M_{f, i_1, \dots, i_n}$$

for $c < 0$

shall give the proof.

Details are omitted here.

$$\text{iv). } f \leq g \Rightarrow m_{f, i \dots i} \leq M_{g, i \dots i}$$

$$\Rightarrow L(f, P) \leq U(g, P)$$

for any partition.

And thus (Since f, g are integrable)

$$\int_A f \leq \int_A g$$

#

Remark on the additional question.

A function is integrable \Leftrightarrow

$$\sup_P L(f, P) = \inf_P U(f, P)$$
$$= \int f \quad (A)$$

where the inf & sup is taken over all possible partitions.

This is equal to say: (B) \exists a partition sequence $\{P_n\}$, where P_{n+1} is the refinement of P_n , the limit

$$\lim_{n \rightarrow \infty} L(f, P_n), \quad \lim_{n \rightarrow \infty} U(f, P_n)$$

exist and both equal to the integral $\int f$.

(A) \Rightarrow (B).

From (A), $\forall n > 0, \exists$ a partition \tilde{P}_n

$$\frac{1}{2n} + L(f, \tilde{P}_n) \geq \sup L(f, P) = \inf U(f, P) \geq U(f, \tilde{P}_n) - \frac{1}{2n}$$

$$\text{i.e. } U(f, \tilde{P}_n) - L(f, \tilde{P}_n) \leq \frac{1}{n}. \text{ and } \lim_{n \rightarrow \infty} U(f, \tilde{P}_n) = \lim_{n \rightarrow \infty} L(f, \tilde{P}_n) = \int f.$$

Let P_n be the refinement of $\hat{P}_n, \hat{P}_{n-1}, \dots, \hat{P}_1, \hat{P}_0$.
 Then $L(f, P_n) \geq L(f, \hat{P}_n)$

$$U(f, P_n) - L(f, P_n) \leq U(f, \hat{P}_n) - L(f, \hat{P}_n) \leq \frac{1}{n}$$

$\underbrace{U(f, P_n) \leq U(f, \hat{P}_n)}_{P_n \text{ is refinement of } \hat{P}_n} \uparrow$

Therefore $\lim_{n \rightarrow \infty} U(f, P_n) = \lim_{n \rightarrow \infty} L(f, P_n)$
 where the existence of limit follows easily
 from

$L(f, P_0) \leq L(f, P_n) \leq U(f, P_n) \leq U(f, P_0)$
 and the monotonicity of upper and lower sum under refinement.

It is clear P_{n+1} is the refinement of P_n
 Moreover. $\lim_{n \rightarrow \infty} L(f, P_n) \geq \lim_{n \rightarrow \infty} L(f, \hat{P}_n) = \int A$
 $\lim_{n \rightarrow \infty} U(f, P_n) \leq \lim_{n \rightarrow \infty} U(f, \hat{P}_n) = \int A$.

(B) \Rightarrow (A) easily from the fact

$$\inf_P U(f, P) < U(f, P_n)$$

$$\sup_P L(f, P) > L(f, P_n)$$

⊗

therefore.

$$0 < \inf_P U(f, P) - \sup_P L(f, P) \leq U(f, P_n) - L(f, P_n)$$

$\longrightarrow 0 \text{ as } n \rightarrow +\infty$

taking $n \rightarrow \infty$ in ⊗ to get (A).